Standard Young Tableaux and the Hook Length Formula

A combinatorial excursion

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Representation Theory of the Symmetric Group Seminar
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Contents

The reference text for this talk is [Lor18, Chapter 4]

1 Hook-Length Formula

1.1 Historical Background: An amusing tale

TABLE 1 Timeline

1900-02	Frobenius-Young determinantal formula for f^λ introduced: **Theorem 1.1** (Determinantal Formula)**.** *For* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ *a partition of* n*, we have* $$f^\lambda = n! \left\| \frac{1}{(\lambda_i - i + j)!} \right\|,$$ *where this determinant is* $m \times m$*.*
1953	Robinson, Frame, and Thrall derive the hook-length formula for f^λ: **Theorem 1.2** (Hook-Length Formula)**.** *For* λ *a partition of* n*, we have* $$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)}.$$
1976	Hillman and Grassl provide first proof of the hook-length formula which leverages the hooks.
1979	Greene, Nijenhuis, and Wilf develop a probabilistic proof leveraging hook walks (see [GNW79]).
1980	Sagan applies the probabilistic algorithm of Greene, Nijenhuis, and Wilf to shifted Young tableaux via shifted hook walks (see [Sag80]).
1982	Remmel, Franzblau and Zeilberger discover bijective proofs.
1997	A direct bijective proof via Schützenberger's *jeu de taquin* algorithm presented by Novelli, Pak, and Stoyanovskii (see [NPS97]).

This celebrated result has since found numerous applications across areas such as algebraic geometry, probability, analysis, and algorithms.

See [Sag80] for an inductive derivation of 1.2 from 1.1.

1.2 Standard Young Tableaux and Paths in $\mathbb{Y}$

Let $\mathbb{Y}$ denote the *Young graph* and $\mathbb{B}$ denote the *Branching graph* of the symmetric group $\mathbb{S}_n$.

Definition 1.2.1. A *standard Young tableaux* of *shape* λ (or a λ*-tableaux*) is obtained by filling a Young diagram of λ with $1, 2, \ldots, n$ such that rows/columns are increasing.

The process of successively removing boxes containing the highest number yields a bijection:

$$\{\text{paths } \square \to \cdots \to \lambda \text{ in } \mathbb{Y}\} \xleftrightarrow{\sim} \{\lambda - \text{tableaux}\}.$$

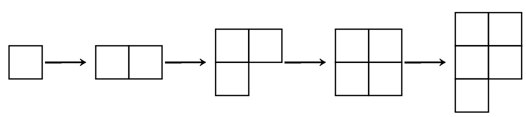
Consequently, we will denote

$$f^\lambda := \#\{\lambda - \text{tableaux}\} = \#\{\text{paths } \square \to \cdots \to \lambda \text{ in } \mathbb{Y}\} = \dim V^\lambda, \tag{1}$$

where $V^\lambda \in \operatorname{Irr} \mathbb{S}_n$ is the irreducible representation associated to λ under the isomorphism $\mathbb{Y} \xrightarrow{\sim} \mathbb{B}$.

Example 1.2.2. We compute all the standard Young tableaux of shape $\lambda = (2, 2, 1)$ with their associated paths in $\mathbb{Y}$.

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$



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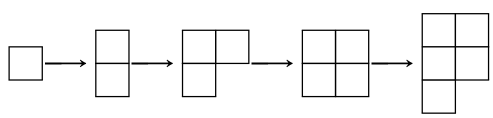

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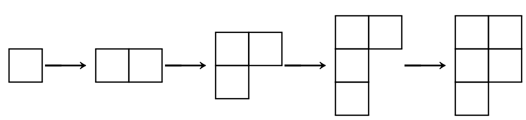

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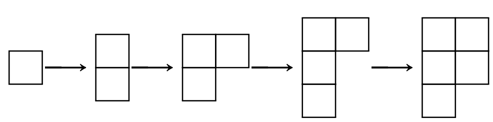

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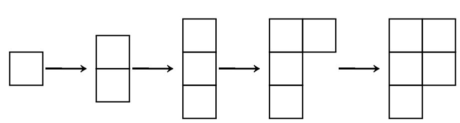


1.3 An experiment: Hook-Walks

Let λ be given by its Young diagram.

Definition 1.3.1. The *hook* at position x of λ consists of the boxes lying to the right and below x.

Definition 1.3.2. The *hook length* at x is given by

$$h(x) := \#\{\text{boxes in the hook at } x\}.$$

It is clear that x is removable if and only if $h(x) = 1$. We now introduce the Hook-Length Formula, the proof of which we assemble in the remainder of the talk.

Theorem 1.2 (Hook-Length Formula)**.** *For λ a partition of n, we have*

$$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)}.$$

To penetrate this simple product formula, we avail ourselves of an example.

4	2
3	1
1	

Consider the partition $\lambda = (2,2,1)$. We fill each box with its respective hook length to obtain the following Young diagram. According to Theorem 1.2,

$$f^\lambda = \frac{5!}{4\cdot 3\cdot 2\cdot 1^2} = 5,$$

which coincides with previous observations.

We now describe an experiment with which we will use to prove Theorem 1.2. In particular, we will consider a memoryless random walk ω on a Young diagram. To begin, consider a partition λ, and choose a box $x = x_0$ uniformly at random (that is, with probability $\frac{1}{n}$). If x is a removable box, stop; otherwise, select a box $x_1 \neq x$ from the hook at x uniformly at random (with probability $\frac{1}{h(x)-1}$). If x_1 is a removable box, stop; otherwise, repeat the above procedure. Since λ is finite, this process terminates after finitely many steps, and returns a walk

$$\omega : x = x_0 \to x_1 \to \cdots \to x_t = c$$

for c removable. We call ω a *hook walk*, and our sample space Ω consisting of all hook walks ω of λ is finite. Then the probability of a given hook walk ω is

$$p(\omega) = \frac{1}{n} q(\omega), \text{ for } q(\omega) = q(x_0)q(x_1)\cdots q(x_{t-1}).$$

For each removable box c, denote $E_c = \{\text{hook walks where } x_t = c\}$. As these events partition our same set, we obtain

$$1 = \sum_{\omega\in\Omega} p(\omega) = \sum_c P(E_c) = \sum_c \frac{1}{n} \sum_{\omega\in E_c} q(\omega). \tag{2}$$

1.4 A Probabilistic Proof

We are now equipped to prove Theorem 1.2.

Proof of Theorem 1.2. Setting $H(\lambda) := \prod_{x\in\lambda} h(x)$, it suffices to show

$$f^\lambda = \frac{n!}{H(\lambda)}.$$

To this end, we perform an induction on n. The statement is clear when $n = 1$.

Now, let $n > 1$. Recall that each path $\square \to \cdots \to \lambda$ restricts to a path $\square \to \cdots \to \mu$ for some $\mu \to \lambda$; conversely, each path $\square \to \cdots \to \mu$ for $\mu \to \lambda$ "lifts" to a unique path to λ. By $\mu \to \lambda$, we mean μ is obtained from λ by deleting a removable box c, and each μ arises uniquely in this way. Thus, we may denote $\mu := \lambda \setminus c$. These observations yields the following recursion:

$$f^\lambda = \sum_c f^{\lambda\setminus c}, \tag{3}$$

where c ranges over all removable boxes. By our induction hypothesis, we have

$$f^{\lambda\setminus c} = \frac{(n-1)!}{H(\lambda\setminus c)}.$$

Recasting (3), we may equivalently show

$$\frac{n!}{H(\lambda)} = \sum_c \frac{(n-1)!}{H(\lambda\setminus c)} \iff 1 = \sum_c \frac{1}{n}\frac{H(\lambda)}{H(\lambda\setminus c)}, \tag{4}$$

where c ranges over all removable boxes. Via (2), we recast once more, and now aim to show

$$\sum_{\omega\in E_c} q(\omega) = \frac{H(\lambda)}{H(\lambda\setminus c)}. \tag{5}$$

We now consider the left- and right-hand sides of (5) separately. For the right-hand side, notice that the only hooks of $\lambda\setminus c$ which differ from those of λ are those at those x lying in the same row or column as c (excluding c itself). Let $\mathbf{B}$ denote this region, which we mark in ▇ in Figure 1. These hooks have length exactly 1 less in $\lambda\setminus c$. We may therefore reformulate the right-hand side as,

$$\frac{H(\lambda)}{H(\lambda\setminus c)} = \prod_{b\in\mathbf{B}} \frac{h(b)}{h(b)-1}.$$

Recalling that $q(b) = \frac{1}{h(b)-1}$ from Section 1.3, we get $\prod_{b\in\mathbf{B}} \frac{h(b)}{h(b)-1} = \prod_{b\in\mathbf{B}}(1+q(b))$. Thus, we recast our right-hand side once more and obtain,

$$\frac{H(\lambda)}{H(\lambda\setminus c)} = \prod_{b\in\mathbf{B}}(1+q(b)) = \sum_{S\subset\mathbf{B}}\prod_{b\in S} q(b). \tag{6}$$

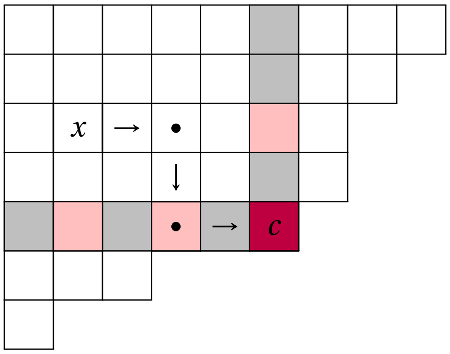


Figure 1: A hook walk $\omega \in E_c$

Regarding the left-hand side, fix a hook walk $\omega \in E_c$ (see $x \to \cdots \to c$ in Figure 1). We denote by $S_\omega \subset \mathbf{B}$ the set of boxes which arise as the horizontal and vertical projections of the boxes of ω into $\mathbf{B}$ (see in Figure 1.) Observe that these projections always determine the starting point x. If it further happens that $x \in \mathbf{B}$, then these projections determine the entire hook walk ω as well.

If it happens that

$$\sum_{\substack{\omega \in E_c \\ S_\omega = S}} q(\omega) = \prod_{b \in S} q(b), \tag{7}$$

then we would obtain

$$\sum_{\omega \in E_c} q(\omega) = \sum_{S \subset \mathbb{B}} \sum_{\substack{\omega \in E_c \\ S_\omega = S}} q(\omega) \overset{(7)}{=} \sum_{S \subset \mathbf{B}} \prod_{b \in S} q(b) \overset{(6)}{=} \frac{H(\lambda)}{H(\lambda \setminus c)},$$

as desired.

Thus, it remains to establish (7). To this end, we argue by induction on $|S|$, $S \subset \mathbf{B}$. The claim is clear when $|S| = 0, 1$, as these correspond to the hook walks ω where $x = c$ or $x \in \mathbf{B}$, respectively. In both cases the left- and right-hand sides of (7) consist of one term. So let $x \notin \mathbf{B} \cup \{c\}$.

Let x_H, x_V denote the horizontal and vertical projections of x into $\mathbf{B}$, respectively. Any hook walk $\omega \in E_c$ achieving $S_\omega = S$ starts either with a move down or a move to the right. For η the remainder of the walk after x, we have $S_\eta = S \setminus \{x_H\}$ in the former case, and $S_\eta = S \setminus \{x_V\}$ in the latter. Quickly note by Figure 2 that,

$$h(x) + 1 = h(x_H) + h(x_V).$$

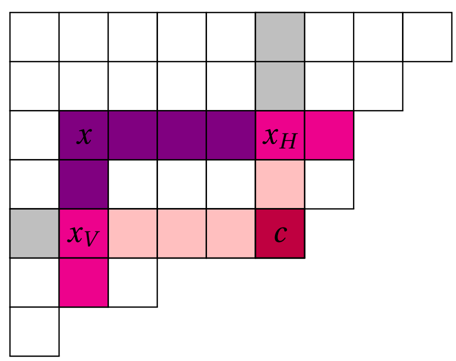


Figure 2: Illustrating Equation (8)

From this it follows that

$$1 = q(x)\left(\frac{1}{q(x_H)} + \frac{1}{q(x_V)}\right). \tag{8}$$

We therefore deduce,

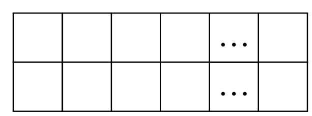
$$\begin{aligned}
\sum_{\substack{\omega\in E_c\\ S\omega=S}} q(\omega) &= q(x)\left(\sum_{\substack{\eta\in E_c\\ S\eta=S\setminus\{x_H\}}} q(\eta)+\sum_{\substack{\eta\in E_c\\ S\eta=S\setminus\{x_V\}}} q(\eta)\right)\\
&\overset{\text{I.H.}}{=} q(x)\left(\prod_{b\in S\setminus\{x_H\}} q(b)+\prod_{b\in S\setminus\{x_V\}} q(b)\right)\\
&= q(x)\left(\frac{1}{q(x_H)}+\frac{1}{q(x_V)}\right)\prod_{b\in S} q(b)\\
&\overset{(8)}{=} \prod_{b\in S} q(b),
\end{aligned}$$

which completes the proof of the hook-length formula. □

Exercise 1.4.1 (Rectangle Partitions)**.** *Show that $f^{(n,n)}=\frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number*

Proof. Let R_1, R_2 denote the first and second rows of the Young diagram of (n,n), respectively. Then,

$$\prod_{x\in(n,n)} h(x)=\left(\prod_{x\in R_1} h(x)\right)\cdot\left(\prod_{x\in R_2} h(x)\right)=(n+1)!\cdot n!=\frac{1}{n+1}n!n!$$



It follows by 1.2 that $f^{(n,n)}=\frac{(2n)!}{(n+1)n!n!}=\frac{1}{n+1}\binom{2n}{n}$. □

**See [Lor18, Exercise 4.3.3]*

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